

Robust \mathcal{H}_∞ Filtering for Markovian Jump Systems With Randomly Occurring Nonlinearities and Sensor Saturation: The Finite-Horizon Case

Hongli Dong, *Member, IEEE*, Zidong Wang, *Senior Member, IEEE*, Daniel W. C. Ho, *Senior Member, IEEE*, and Huijun Gao, *Senior Member, IEEE*

Abstract—This paper addresses the robust \mathcal{H}_∞ filtering problem for a class of discrete time-varying Markovian jump systems with randomly occurring nonlinearities and sensor saturation. Two kinds of transition probability matrices for the Markovian process are considered, namely, the one with polytopic uncertainties and the one with partially unknown entries. The nonlinear disturbances are assumed to occur randomly according to stochastic variables satisfying the Bernoulli distributions. The main purpose of this paper is to design a robust filter, over a given finite-horizon, such that the \mathcal{H}_∞ disturbance attenuation level is guaranteed for the time-varying Markovian jump systems in the presence of both the randomly occurring nonlinearities and the sensor saturation. Sufficient conditions are established for the existence of the desired filter satisfying the \mathcal{H}_∞ performance constraint in terms of a set of recursive linear matrix inequalities. Simulation results demonstrate the effectiveness of the developed filter design scheme.

Index Terms—Discrete time-varying systems, Markovian jumping parameters, randomly occurring nonlinearities, robust \mathcal{H}_∞ filtering, sensor saturation.

I. INTRODUCTION

AS a class of hybrid system, Markovian jump systems (MJSs) have been attracting extensive research attention in the past years since MJSs are very appropriate to model the dynamic systems whose structure is subject to random abrupt variation mainly due to, for example, component failures or

repairs, changing subsystem interconnections, sudden environmental disturbance and abrupt variations of the operating points of a nonlinear system; see, e.g., [12]–[14], [18] and the references therein. In most existing literature, the transition probabilities in the jumping process, which determine the system behavior to a great extent, have been assumed to be completely accessible. However, such an ideal assumption would inevitably limit the application of established results because of the difficulty and cost in obtaining precisely all the transition probabilities. Very recently, some initial results have been obtained in [22], [23], [29], and [30] for Markovian jumping systems with *partially* unknown transition probabilities.

For decades, filtering technique has been playing an important role in a variety of application areas including target tracking, image processing, signal processing and control engineering [14], [25], [28]. Among other existing filtering methods, the \mathcal{H}_∞ filtering approach is closely related to many robustness problems such as stabilization and sensitivity minimization of uncertain systems, and has therefore gained persistent attention. Recently, much progress has been made in the study of the \mathcal{H}_∞ filtering problem for Markovian jump systems; see, e.g., [1], [6], [7], [18], [20], and [21]. In particular, the \mathcal{H}_∞ filters have been designed in [1] for Markovian jump linear systems and in [2] and [31] for nonlinear systems. When both the Markovian jump parameters and time-delays appear in the systems, the \mathcal{H}_∞ filtering problems have been studied in [6], [12], and [13]. In [18], a delay-dependent approach has been developed to deal with the stochastic \mathcal{H}_∞ filtering problem for a class of stochastic time-delay jumping systems subject to both sensor nonlinearities and exogenous nonlinear disturbances. It is noted that a common feature of the aforementioned \mathcal{H}_∞ filtering results is that the complete knowledge is required for the transition probabilities of the jump process. Very recently, the problem of \mathcal{H}_∞ filtering has been investigated in [30] for a class of discrete-time Markovian jump linear systems with partly unknown transition probabilities.

On another research front, networked control systems (NCSs) have become more and more popular for their successful industrial applications in aircrafts, manufacturing plants, automobiles, etc. The issues of random data packet dropouts [3], [8], [15], [20], random transmission delay [4], [9], [27], and signal quantization [11], which typically emerge in NCSs, have been well studied in the literature. Nevertheless, the randomly occurring nonlinearities (RONs), which also constitute an important class of network-induced phenomena, have been largely

Manuscript received February 01, 2010; revised September 06, 2010 and February 12, 2011; accepted March 18, 2011. Date of publication April 05, 2011; date of current version June 15, 2011. The associate editor coordinating the review of this manuscript and approving it for publication was Dr Suleyman S. Kozat. This work was supported in part by the National Natural Science Foundation of China under Grants 61028008, 60825303, and 61004067, National 973 Project under Grant 2009CB320600, the Key Laboratory of Integrated Automation for the Process Industry (Northeastern University) from the Ministry of Education of China, the Engineering and Physical Sciences Research Council (EPSRC) of the U.K., under Grant GR/S27658/01, the Royal Society of the U.K., and the Alexander von Humboldt Foundation of Germany.

H. Dong is with the Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China, and also with the College of Electrical and Information Engineering, Northeast Petroleum University, Daqing 163318, China.

Z. Wang is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, U.K. (e-mail: Zidong.Wang@brunel.ac.uk).

D. W. C. Ho is with Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong.

H. Gao is with the Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China.

Color versions of one or more of the figures in this paper are available online at <http://ieeexplore.ieee.org>.

Digital Object Identifier 10.1109/TSP.2011.2135854

overlooked. As is well known, nonlinearities are widespread in practice, most of which can be described as the additive nonlinear disturbances caused by environmental circumstances. In the case of NCSs with communication constraints, such nonlinear disturbances may occur in a probabilistic way, that is, they may be randomly changeable in terms of their types and/or intensity. This gives rise to the so-called RONS that have started to receive some initial research interests in, for instance, [17] for complex networks.

It has been a well-recognized fact that, virtually, almost all models for real-time systems behave in a time-varying way, especially those after digital discretization. As such, time-varying stochastic systems have recently been paid much research attention; see, e.g., [5], [26], [28], [32] and the references therein. Unfortunately, in most existing literature, it is implicitly assumed that the measurements are always working under the linear condition such that the possible effect of amplitude saturation is ignored [10], [16], [19], [24]. Such an assumption is obviously unrealistic as sensors are all subject to physical saturations. Therefore, sensor saturation issue should have taken into account when designing filters. It should be pointed out that the robust \mathcal{H}_∞ filtering problem for time-varying stochastic systems with sensor saturation is still an open and challenging issue, not to mention the case where the inaccessible Markovian jump parameter and RONS are also involved over a finite horizon. It is, therefore, the purpose of this paper to shorten such a gap.

The main contributions of this paper are listed as follows. 1) A distinguishing feature of the research problem addressed is that the time-varying system is considered and, accordingly, the finite-horizon filtering problem is investigated by developing an effective recursive linear matrix inequalities (RLMIs) approach. 2) In the plant under consideration, Markovian jumping parameters, time-varying parameters, randomly occurring nonlinearities, sensor saturation as well as parameter uncertainties exist simultaneously, which render more practical significance of our current research. Note that the sensor saturation issue has seldom been taken into account in filter design. 3) Randomly occurring nonlinearities are introduced to describe the phenomena of nonlinear disturbances appearing in a random way. 4) Two cases are considered for the transition probability matrix of the Markovian process with either polytopic uncertainties or partially unknown entries. 5) The proposed robust H_∞ filtering technique is dependent not only on the current available state estimate but also on the previous measurement, which serves as a recursive algorithm suitable for online application.

Notation: The notation used in the paper is fairly standard. The notation $\text{diag}\{\dots\}$ stands for a block-diagonal matrix. $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ will, respectively, mean expectation of x and expectation of x conditional on y . $\text{Prob}\{\cdot\}$ represents the occurrence probability of the event “ \cdot ”.

II. PROBLEM FORMULATION

Let $r(k)$ ($k \in [0, N]$) be a Markov chain taking values in a finite state space $S = \{1, 2, \dots, s\}$ with transition probability matrix $\hat{\Psi} = [\lambda_{ij}]$ given by

$$\text{Prob}\{r(k+1) = j | r(k) = i\} = \lambda_{ij}, \quad \forall i, j \in S$$

where $\lambda_{ij} \geq 0$ ($i, j \in S$) is the transition probability from i to j and $\sum_{j=1}^s \lambda_{ij} = 1, \forall i \in S$.

In this paper, we consider the following two cases where the transition probability matrix $\hat{\Psi} = [\lambda_{ij}]$ is imperfectly known.

Case 1: The transition probability matrix $\hat{\Psi}$ belongs to a given polytope, namely, $\hat{\Psi} \in \mathfrak{R}$, where \mathfrak{R} is a given convex bounded polyhedral domain described by ν vertices as follows:

$$\mathfrak{R} := \left\{ \hat{\Psi} \left| \hat{\Psi} = \sum_{r=1}^{\nu} \psi_r \hat{\Psi}^{(r)}, \sum_{r=1}^{\nu} \psi_r = 1, \psi_r \geq 0, \right. \right. \\ \left. \left. r = 1, 2, \dots, \nu \right\} \quad (1)$$

and $\hat{\Psi}^{(r)} = [\lambda_{ij}^{(r)}]$ ($i, j = 1, \dots, s, r = 1, \dots, \nu$) are given transition probability matrices. It is easy to see that the convex combination of these transition probability matrices is also a possible transition probability matrix.

Case 2: Some elements in matrix $\hat{\Psi}$ are unknown, for example, the transition probability matrix $\hat{\Psi}$ may be

$$\hat{\Psi} = \begin{bmatrix} \lambda_{11} & ? & ? \\ ? & \lambda_{22} & ? \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

where “?” represents the unknown entries. For notation clarity, for any $i \in S$, we denote that

$$\hat{\Psi}_k^i := \{j : \lambda_{ij} \text{ is known}\} \\ \hat{\Psi}_{uk}^i := \{j : \lambda_{ij} \text{ is unknown}\}.$$

In this paper, we consider the following class of uncertain discrete stochastic nonlinear time-varying Markovian jump systems in the presence of sensor saturation defined on $k \in [0, N]$:

$$\begin{cases} x(k+1) = (A(k, r(k)) + \Delta A(k, r(k)))x(k) \\ \quad + \alpha(k)f(k, x(k)) + D_1(k, r(k))w(k) \\ y(k) = \sigma(y_s(k, r(k))) + \beta(k)g(k, x(k)) \\ \quad + D_2(k, r(k))w(k) \\ z(k) = (L(k, r(k)) + \Delta L(k, r(k)))x(k) \\ y_s(k, r(k)) = C(k, r(k))x(k) \\ x(0) = \varphi_0 \end{cases} \quad (2)$$

where $x(k) \in \mathbb{R}^n$ represents the state vector; $y(k) \in \mathbb{R}^r$ is the measurement output; $z(k) \in \mathbb{R}^m$ is a linear combination of the state variables to be estimated; $w(k) \in \mathbb{R}^p$ is the disturbance input which belongs to $l_2[0, N]$, φ_0 is a given real initial value. For the fixed system mode, $A(k, r(k)), D_1(k, r(k)), C(k, r(k)), D_2(k, r(k)), L(k, r(k))$ are known, real, time-varying matrices with appropriate dimensions. $\Delta A(k, r(k))$ and $\Delta L(k, r(k))$ are unknown matrices representing the time-varying parameter uncertainties of the form

$$\begin{bmatrix} \Delta A(k, r(k)) \\ \Delta L(k, r(k)) \end{bmatrix} = \begin{bmatrix} H_1(r(k)) \\ H_2(r(k)) \end{bmatrix} F(k, r(k)) N(r(k))$$

where, for fixed system mode, $H_1(r(k)), H_2(r(k))$, and $N(r(k))$ are known, real, constant matrices of appropriate dimensions which characterize how the uncertain parameter in $F(k, r(k))$ enters the nominal matrices $A(k, r(k))$ and

$L(k, r(k))$, and $F(k, r(k))$ is an unknown time-varying matrix satisfying

$$F^T(k, r(k))F(k, r(k)) \leq I, \quad \forall r(k) \in S. \quad (3)$$

The nonlinear functions $f(k, x(k))$ and $g(k, x(k))$ satisfy the following conditions:

$$\begin{aligned} \|f(k, x(k))\|^2 &\leq \varepsilon_1(k) \|E_1(k)x(k)\|^2 \\ \|g(k, x(k))\|^2 &\leq \varepsilon_2(k) \|E_2(k)x(k)\|^2 \end{aligned} \quad (4)$$

for all $k \in [0, N]$, where $\varepsilon_1(k) > 0$ and $\varepsilon_2(k) > 0$ are known positive scalars, and $E_1(k)$ and $E_2(k)$ are known constant matrices.

The stochastic variables $\alpha(k)$ and $\beta(k)$ are two independent Bernoulli sequences which account for the phenomena of randomly occurred nonlinearities. A natural assumption on the sequences $\alpha(k)$ and $\beta(k)$ can be made as follows:

$$\begin{aligned} \text{Prob}\{\alpha(k) = 1\} &= \bar{\alpha}, \quad \text{Prob}\{\alpha(k) = 0\} = 1 - \bar{\alpha}, \\ \text{Prob}\{\beta(k) = 1\} &= \bar{\beta}, \quad \text{Prob}\{\beta(k) = 0\} = 1 - \bar{\beta} \end{aligned}$$

where $\bar{\alpha} \in [0, 1]$ and $\bar{\beta} \in [0, 1]$ are known constants. We also assume that the $r(k)$, $\alpha(k)$ and $\beta(k)$ are mutually independent.

Remark 1: As described in (2), the nonlinear functions $f(k, x(k))$ and $g(k, x(k))$ could occur independently and randomly according to individual probability distributions specified *a priori* through statistical tests.

The saturation function $\sigma(\cdot) : \mathbb{R}^r \mapsto \mathbb{R}^r$ is defined as

$$\sigma(v) = [\sigma_1^T(v_1) \quad \sigma_2^T(v_2) \quad \cdots \quad \sigma_r^T(v_r)]^T \quad (5)$$

with $\sigma_i(v_i) = \text{sign}(v_i) \min\{V_{i,\max}, |v_i|\}$, where $V_{i,\max}$ is the i th element of the vector V_{\max} , the saturation level.

Definition 1: [24] A nonlinearity $\Psi : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to satisfy a sector condition if

$$(\Psi(v) - \bar{H}_1 v)^T (\Psi(v) - \bar{H}_2 v) \leq 0, \quad \forall v \in \mathbb{R}^r \quad (6)$$

for some real matrices $\bar{H}_1, \bar{H}_2 \in \mathbb{R}^{r \times r}$, where $\bar{H} = \bar{H}_2 - \bar{H}_1$ is a positive-definite symmetric matrix. In this case, we say that Ψ belongs to the sector $[\bar{H}_1 \bar{H}_2]$.

If we assume that there exist two diagonal matrices K_1 and K_2 such that $0 \leq K_1 < I \leq K_2$, then the saturation function $\sigma(y_s(k, r(k)))$ in (2) can be decomposed into a linear and a nonlinear part as

$$\sigma(y_s(k, r(k))) = K_1 C(k, r(k))x(k) + \Psi(y_s(k, r(k))) \quad (7)$$

where $\Psi(y_s(k, r(k)))$ is a nonlinear vector-valued function satisfying a sector condition with $\bar{H}_1 = 0$, $\bar{H}_2 = K$, and can be described as follows:

$$\Psi^T(y_s(k, r(k))) (\Psi(y_s(k, r(k))) - Ky_s(k, r(k))) \leq 0 \quad (8)$$

where $K = K_2 - K_1$.

In this paper, the linear time-varying filter under consideration is of the following structure:

$$\begin{cases} \hat{x}(k+1) = A_f(k, r(k))\hat{x}(k) + B_f(k, r(k))y(k) \\ \hat{z}(k) = L_f(k, r(k))\hat{x}(k) \end{cases} \quad (9)$$

where $\hat{x}(k) \in \mathbb{R}^n$ represents the state estimate and $z(k) \in \mathbb{R}^m$ is the estimated output. For fixed system mode, the time-varying matrices $A_f(k, r(k))$, $B_f(k, r(k))$ and $L_f(k, r(k))$ are the filter parameters to be designed.

For presentation convenience, for each possible $r(k) = i$ ($i \in S$), a matrix $N(k, r(k))$ will be denoted by $N_i(k)$.

Let us now work on the system mode $r(k) = i, \forall i \in S$. Setting $\bar{x}(k) = [x^T(k) \quad \hat{x}^T(k)]^T$ and $\bar{z}(k) = z(k) - \hat{z}(k)$, we obtain an augmented system from (2) and (9) as follows:

$$\begin{cases} \bar{x}(k+1) = \bar{A}_i(k)\bar{x}(k) + \bar{B}_i(k)\Psi(y_{si}(k)) \\ \quad + (G_i(k) + \bar{G}_i(k))h(k, x(k)) + \bar{D}_i(k)w(k) \\ \bar{z}(k) = \bar{L}_i(k)\bar{x}(k) \end{cases} \quad (10)$$

where

$$\begin{aligned} \bar{A}_i(k) &= \begin{bmatrix} A_i(k) + \Delta A_i(k) & 0 \\ B_{fi}(k)K_1C_i(k) & A_{fi}(k) \end{bmatrix} \\ \bar{B}_i(k) &= \begin{bmatrix} 0 \\ B_{fi}(k) \end{bmatrix}, \bar{D}_i(k) = \begin{bmatrix} D_{1i}(k) \\ B_{fi}(k)D_{2i}(k) \end{bmatrix} \\ G_i(k) &= \text{diag}\{\bar{\alpha}I, \bar{\beta}B_{fi}(k)\} \\ \bar{G}_i(k) &= \text{diag}\{(\alpha(k) - \bar{\alpha})I, (\beta(k) - \bar{\beta})B_{fi}(k)\} \\ h(k, x(k)) &= \begin{bmatrix} f(k, x(k)) \\ g(k, x(k)) \end{bmatrix} \\ \bar{L}_i(k) &= [L_i(k) + \Delta L_i(k) \quad -L_{fi}(k)]. \end{aligned} \quad (11)$$

Our aim in this paper is to design a finite-horizon filter in the form of (9) such that, for the given disturbance attenuation level $\gamma > 0$, positive definite matrices Q_i ($i = 1, 2, \dots, s$) and the initial state $\bar{x}(0)$, the \mathcal{H}_∞ performance index satisfies the following inequality:

$$\mathbb{E} \left\{ \|\bar{z}(k)\|_{[0, N]}^2 \right\} \leq \gamma^2 \left(\mathbb{E} \left\{ \|w(k)\|_{[0, N]}^2 \right\} + e^T(0)Q_i e(0) \right) \quad (12)$$

where $e(0) = x(0) - \hat{x}(0)$.

III. PERFORMANCE ANALYSIS OF ROBUST \mathcal{H}_∞ FILTER

Lemma 1 [24]: Let $Y_0(\eta), Y_1(\eta), \dots, Y_p(\eta)$ be quadratic functions of $\eta \in \mathbb{R}^n$, $Y_i(\eta) = \eta^T T_i \eta$, $i = 0, 1, \dots, p$, with $T_i = T_i^T$. Then, the implication $Y_1(\eta) \leq 0, \dots, Y_p(\eta) \leq 0 \Rightarrow Y_0(\eta) \leq 0$ holds if there exist $\tau_1, \dots, \tau_p > 0$ such that

$$T_0 - \sum_{i=1}^p \tau_i T_i \leq 0. \quad (13)$$

For presentation convenience, we denote

$$\begin{aligned}
\Omega_{11i}(k) &= 2\bar{A}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) - P_i(k) \\
&\quad + \bar{L}_i^T(k)\bar{L}_i(k) + \rho_i(k)\bar{E}(k) \\
\Omega_{21i}(k) &= \bar{B}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) + \frac{1}{2}\tau_i(k)\tilde{C}_i(k) \\
\Omega_{22i}(k) &= 2\bar{B}_i^T(k)\bar{P}_i(k+1)\bar{B}_i(k) - \tau_i(k)I \\
\Omega_{31i}(k) &= \bar{D}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) \\
\Omega_{32i}(k) &= \bar{D}_i^T(k)\bar{P}_i(k+1)\bar{B}_i(k) \\
\Omega_{33i}(k) &= 2\bar{D}_i^T(k)\bar{P}_i(k+1)\bar{D}_i(k) - \gamma^2 I \\
\bar{P}_i(k) &= \sum_{j=1}^s \lambda_{ij} P_j(k), \quad i, j = 1, \dots, s \\
\hat{G}_i(k) &= \text{diag}\{\sqrt{\alpha(1-\alpha)}I, \sqrt{\beta(1-\beta)}B_{fi}(k)\} \\
\tilde{C}_i(k) &= [KC_i(k) \ 0] \\
\bar{E}(k) &= \text{diag}\{E_\varepsilon(k), 0\} \\
E_\varepsilon(k) &= \varepsilon_1(k)E_1^T(k)E_1(k) + \varepsilon_2(k)E_2^T(k)E_2(k). \quad (14)
\end{aligned}$$

Theorem 1: Consider system (2) subject to randomly occurring nonlinearities (4) and sensor saturation (5). Let the disturbance attenuation level $\gamma > 0$, sets of positive scalars $\{\rho_i(k) > 0, i \in S\}_{0 \leq k \leq N}$, $\{\tau_i(k) > 0, i \in S\}_{0 \leq k \leq N}$, positive definite matrices $\bar{Q}_i > 0, i \in S$ and the filter parameters $\{A_{fi}(k)\}_{0 \leq k \leq N}$, $\{B_{fi}(k)\}_{0 \leq k \leq N}$ and $\{L_{fi}(k)\}_{0 \leq k \leq N}$ ($i \in S$) be given. The \mathcal{H}_∞ performance index defined in (12) is achieved for all nonzero $w(k)$ if, with the initial condition $P_i(0) \leq \gamma^2[I - I]^T Q_i [I - I]$, there exists a family of positive definite matrices $\{P_i(k)\}_{0 \leq k \leq N+1}$ ($i \in S$) satisfying the following recursive matrix inequalities:

$$\begin{bmatrix} -\rho_i(k)I & * & * \\ \hat{G}_i(k) & -\bar{P}_i^{-1}(k+1) & * \\ G_i(k) & 0 & -\frac{1}{4}\bar{P}_i^{-1}(k+1) \end{bmatrix} \leq 0 \quad (15)$$

$$\begin{bmatrix} \Omega_{11i}(k) & * & * \\ \Omega_{21i}(k) & \Omega_{22i}(k) & * \\ \Omega_{31i}(k) & \Omega_{32i}(k) & \Omega_{33i}(k) \end{bmatrix} \leq 0 \quad (16)$$

for all $0 \leq k \leq N$, where $\Omega_{11i}(k), \Omega_{21i}(k), \Omega_{22i}(k), \Omega_{31i}(k), \Omega_{32i}(k), \Omega_{33i}(k), \bar{P}_i(k), \hat{G}_i(k), \tilde{C}_i(k), E_\varepsilon(k), \bar{E}(k)$ are defined in (14).

Proof: For $r(k) = i$, define the following Lyapunov function

$$V(\bar{x}(k), r(k)) = \bar{x}^T(k)P_i(k)\bar{x}(k) \quad (17)$$

where $P_i(k) = \text{diag}\{P_{1i}(k), P_{2i}(k)\} > 0$ are the solutions to (15) and (16). Then, for $r(k) = i$ and $r(k+1) = j$, one has from (10) that

$$\begin{aligned}
&\mathbb{E}\{\Delta V(\bar{x}(k), r(k))\} \\
&\leq \mathbb{E}\{\eta^T(k)\Lambda_k\eta(k) - \bar{z}^T(k)\bar{z}(k) + \gamma^2\omega^T(k)\omega(k)\} \quad (18)
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_k &= \begin{bmatrix} \Omega_{11i}(k) & * & * \\ \bar{\Omega}_{21i}(k) & \bar{\Omega}_{22i}(k) & * \\ \Omega_{31i}(k) & \Omega_{32i}(k) & \Omega_{33i}(k) \end{bmatrix} \\
\eta(k) &= [\bar{x}^T(k) \quad \Psi^T(y_{si}(k)) \quad \omega^T(k)]^T \\
\bar{\Omega}_{21i}(k) &= \bar{B}_i^T(k)\bar{P}_i(k+1)\bar{A}_i(k) \\
\bar{\Omega}_{22i}(k) &= 2\bar{B}_i^T(k)\bar{P}_i(k+1)\bar{B}_i(k). \quad (19)
\end{aligned}$$

Hence, the \mathcal{H}_∞ performance index defined in (12) is given by

$$\begin{aligned}
&\mathbb{E}\{\|\bar{z}(k)\|_{[0,N]}^2\} \\
&\quad - \gamma^2 \left(\mathbb{E}\{\|w(k)\|_{[0,N]}^2\} + e^T(0)Q_i e(0) \right) \\
&\leq \mathbb{E}\left\{ \sum_{k=0}^N \eta^T(k)\Lambda_k\eta(k) \right\} \\
&\quad - \mathbb{E}\{\bar{x}^T(N+1)\bar{P}_i(N+1)\bar{x}(N+1)\} \\
&\quad + \bar{x}^T(0)(P_i(0) - \gamma^2[I - I]^T \\
&\quad \times Q_i[I - I])\bar{x}(0). \quad (20)
\end{aligned}$$

Noting that $\bar{P}_i(N+1) > 0$ and the initial condition $P_i(0) \leq \gamma^2[I - I]^T Q_i [I - I]$, we can get (12) when the following inequality holds:

$$\eta^T(k)\Lambda_k\eta(k) \leq 0 \quad (21)$$

In terms of the sensor saturation constraint in (8), we have

$$\Psi^T(y_{si}(k))(\Psi(y_{si}(k)) - \tilde{C}_i(k)\bar{x}(k)) \leq 0 \quad (22)$$

which can be written in $\eta(k)$ as

$$\eta^T(k)\Phi_k\eta(k) \leq 0 \quad (23)$$

where

$$\Phi_k = \frac{1}{2} \begin{bmatrix} 0 & * & * \\ -\tilde{C}_i(k) & 2I & * \\ 0 & 0 & 0 \end{bmatrix}. \quad (24)$$

Now, it suffices to find a condition such that (21) holds subject to the sensor saturation constraints (23). By using Lemma 1, the sufficient condition such that the inequalities (23) imply (21) is that there exist positive scalars $\tau_i(k)$ such that

$$\Lambda_k - \tau_i(k)\Phi_k \leq 0 \quad (25)$$

and then the rest of the proof follows from the statement of Theorem 1 immediately. The proof is complete. ■

IV. DESIGN OF ROBUST \mathcal{H}_∞ FILTERS

In this section, given the imperfect transition probability matrix described in *Case 1* and *Case 2*, we shall discuss the robust

\mathcal{H}_∞ filter design problem for the discrete time-varying Markovian jump systems with randomly occurring nonlinearities and sensor saturation. Before presenting the theorem, let us denote

$$\begin{aligned}
\bar{P}_{1i}^{(r)}(k) &= \sum_{j=1}^s \lambda_{ij}^{(r)} P_{1j}(k) \\
\bar{P}_{2i}^{(r)}(k) &= \sum_{j=1}^s \lambda_{ij}^{(r)} P_{2j}(k) \\
i, j &= 1, \dots, s, r = 1, \dots, \nu. \\
\Phi_{11i}(k) &= \text{diag} \{-\rho_i(k)I, -\rho_i(k)I\} \\
\Phi_{22i}^{(r)}(k) &= \text{diag} \left\{ -\bar{P}_{1i}^{(r)}(k+1), -\bar{P}_{2i}^{(r)}(k+1), \right. \\
&\quad \left. -\frac{1}{4}\bar{P}_{1i}^{(r)}(k+1), -\frac{1}{4}\bar{P}_{2i}^{(r)}(k+1) \right\} \\
\Phi_{21i}^{(r)}(k) &= \begin{bmatrix} \tilde{\alpha}\bar{P}_{1i}^{(r)}(k+1) & 0 \\ 0 & \tilde{\beta}\bar{N}_i^{(r)}(k) \\ \tilde{\alpha}\bar{P}_{1i}^{(r)}(k+1) & 0 \\ 0 & \tilde{\beta}\bar{N}_i^{(r)}(k) \end{bmatrix} \\
\tilde{\alpha} &= \sqrt{\tilde{\alpha}(1-\tilde{\alpha})}, \tilde{\beta} = \sqrt{\tilde{\beta}(1-\tilde{\beta})} \\
\Upsilon_{11i}(k) &= \begin{bmatrix} \Xi_{1i}(k) & * \\ \Xi_{2i}(k) & \Xi_{3i}(k) \end{bmatrix} \\
\Upsilon_{21i}^{(r)}(k) &= \begin{bmatrix} \Xi_{4i}^{(r)}(k) & \Xi_{5i}^{(r)}(k) \\ \Xi_{6i}^{(r)}(k) & 0 \end{bmatrix} \\
\Upsilon_{22i}^{(r)}(k) &= \text{diag} \left\{ -\bar{P}_{1i}^{(r)}(k+1), -\bar{P}_{2i}^{(r)}(k+1) \right. \\
&\quad \left. -\bar{P}_{1i}^{(r)}(k+1), -\bar{P}_{2i}^{(r)}(k+1) -\bar{P}_{1i}^{(r)}(k+1) \right\} \\
\Upsilon_{31i}^{(r)}(k) &= \begin{bmatrix} 0 & \Xi_{7i}^{(r)}(k) \\ \Xi_{8i} & 0 \end{bmatrix} \\
\Upsilon_{32i} &= \begin{bmatrix} 0 & 0 \\ \Xi_{9i} & 0 \end{bmatrix} \\
\Upsilon_{33i}^{(r)}(k) &= \text{diag} \left\{ -\bar{P}_{2i}^{(r)}(k+1), -\bar{P}_{1i}^{(r)}(k+1) \right. \\
&\quad \left. -\bar{P}_{2i}^{(r)}(k+1), -\xi_i I, -\xi_i I \right\} \\
\Xi_{1i}(k) &= \text{diag}\{\Pi_{1i}(k), -P_{2i}(k)\} \\
\Xi_{3i}(k) &= \text{diag}\{-\tau_i(k)I, -\gamma^2 I, -I\} \\
\Pi_{1i}(k) &= -P_{1i}(k) + \rho_i(k) \\
&\quad \times (\varepsilon_1(k)E_1^T(k)E_1(k) + \varepsilon_2(k)E_2^T(k)E_2(k)) \\
\Xi_{2i}(k) &= \begin{bmatrix} \frac{1}{2}\tau_i(k)KC_i(k) & 0 \\ 0 & 0 \\ L_i(k) & -L_{fi}(k) \end{bmatrix} \\
\Xi_{4i}^{(r)}(k) &= \begin{bmatrix} \bar{P}_{1i}^{(r)}(k+1)A_i(k) & 0 \\ \bar{N}_i^{(r)}(k)K_1C_i(k) & \bar{M}_i^{(r)}(k) \end{bmatrix} \\
\Xi_{5i}^{(r)}(k) &= \begin{bmatrix} 0 & \bar{P}_{1i}^{(r)}(k+1)D_{1i}(k) & 0 \\ \bar{N}_i^{(r)}(k) & \bar{N}_i^{(r)}(k)D_{2i}(k) & 0 \end{bmatrix} \\
\Xi_{6i}^{(r)}(k) &= \begin{bmatrix} \bar{P}_{1i}^{(r)}(k+1)A_i(k) & 0 \\ \bar{N}_i^{(r)}(k)K_1C_i(k) & \bar{M}_i^{(r)}(k) \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Xi_{7i}^{(r)}(k) &= \begin{bmatrix} \bar{N}_i^{(r)}(k) & 0 & 0 \\ 0 & \bar{P}_{1i}^{(r)}(k+1)D_{1i}(k) & 0 \\ 0 & \bar{N}_i^{(r)}(k)D_{2i}(k) & 0 \\ 0 & 0 & H_{2i}^T \end{bmatrix} \\
\Xi_{8i} &= [\xi_i N_i \quad 0] \\
\Xi_{9i} &= \begin{bmatrix} H_{1i}^T & 0 & H_{1i}^T \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Theorem 2: Consider system (2) with unknown transition probability matrix described in *Case 1*. Let $\gamma > 0$ be a given disturbance attenuation level. For given positive definite matrices $Q_i > 0$ ($i \in S$), if there exist families of positive definite matrices $\{P_{1i}(k)\}_{0 \leq k \leq N+1}$, $\{P_{2i}(k)\}_{0 \leq k \leq N+1}$ ($i = 1, 2, \dots, s$), families of positive scalars $\{\rho_i(k)\}_{0 \leq k \leq N}$, $\{\tau_i(k)\}_{0 \leq k \leq N}$ ($i = 1, 2, \dots, s$) and $\{\xi_i\}_{i=1,2,\dots,s}$, and families of real-valued matrices $\{\bar{M}_i^{(r)}(k)\}_{0 \leq k \leq N}$, $\{\bar{N}_i^{(r)}(k)\}_{0 \leq k \leq N}$ and $\{L_{fi}(k)\}_{0 \leq k \leq N}$ satisfying the following RLMI:

$$\begin{bmatrix} \Phi_{11i}(k) & * \\ \Phi_{21i}^{(r)}(k) & \Phi_{22i}^{(r)}(k) \end{bmatrix} < 0, \quad i = 1, 2, \dots, s, r = 1, 2, \dots, \nu. \quad (26)$$

$$\begin{bmatrix} \Upsilon_{11i}(k) & * & * \\ \Upsilon_{21i}^{(r)}(k) & \Upsilon_{22i}^{(r)}(k) & * \\ \Upsilon_{31i}^{(r)}(k) & \Upsilon_{32i} & \Upsilon_{33i}^{(r)}(k) \end{bmatrix} \leq 0, \quad i = 1, 2, \dots, s, r = 1, 2, \dots, \nu \quad (27)$$

with the initial condition

$$\begin{bmatrix} P_{1i}(0) - \gamma^2 Q_i & \gamma^2 Q_i \\ \gamma^2 Q_i & P_{2i}(0) - \gamma^2 Q_i \end{bmatrix} \leq 0 \quad (28)$$

where $\bar{P}_{1i}^{(r)}(k)$, $\Phi_{11i}(k)$, $\Phi_{22i}^{(r)}(k)$, $\Phi_{21i}^{(r)}(k)$, $\tilde{\alpha}$, $\Upsilon_{11i}(k)$, $\Upsilon_{21i}^{(r)}(k)$, $\Upsilon_{22i}^{(r)}(k)$, $\Upsilon_{31i}^{(r)}(k)$, Υ_{32i} , $\Upsilon_{33i}^{(r)}(k)$, $\Xi_{1i}(k)$, $\Xi_{3i}(k)$, $\Pi_{1i}(k)$, $\Xi_{2i}(k)$, $\Xi_{4i}^{(r)}(k)$, $\Xi_{5i}^{(r)}(k)$, $\Xi_{6i}^{(r)}(k)$, $\Xi_{7i}^{(r)}(k)$, Ξ_{8i} and Ξ_{9i} are defined previously, then there exists an n th-order filter of the form (9) which ensures the \mathcal{H}_∞ performance constraint in (12), where $L_{fi}(k)$ is given as part of the RLMI solution and the other two filter parameters are given by

$$\begin{aligned}
A_{fi}(k) &= \bar{P}_{2i}^{(r)-1}(k+1)\bar{M}_i^{(r)}(k) \\
B_{fi}^{(r)}(k) &= \bar{P}_{2i}^{(r)-1}(k+1)\bar{N}_i^{(r)}(k).
\end{aligned}$$

Proof: Since the transition probability matrix $\hat{\Psi} = [\lambda_{ij}]$ belongs to the convex polyhedral set \mathfrak{R} , there always exist scalars $\psi_r \geq 0$ ($r = 1, 2, \dots, \nu$) such that $\hat{\Psi} = \sum_{r=1}^\nu \psi_r \hat{\Psi}^{(r)}$, $\sum_{r=1}^\nu \psi_r = 1$, where $\hat{\Psi}^{(r)} = [\lambda_{ij}^{(r)}]$ ($r = 1, 2, \dots, \nu$) are ν vertexes of the polytope. Hence, considering (15) and (16) in Theorem 1, we have

$$\begin{bmatrix} -\rho_i(k)I & * & * \\ \hat{G}_i(k) & -\bar{P}_i^{(r)-1}(k+1) & * \\ G_i(k) & 0 & -\frac{1}{4}\bar{P}_i^{(r)-1}(k+1) \end{bmatrix} \leq 0 \quad (29)$$

$$\begin{bmatrix} \Omega_{11i}^{(r)}(k) & * & * \\ \Omega_{22i}^{(r)}(k) & \Omega_{22i}^{(r)}(k) & * \\ \Omega_{31i}^{(r)}(k) & \Omega_{32i}^{(r)}(k) & \Omega_{33i}^{(r)}(k) \end{bmatrix} \leq 0 \quad (30)$$

where

$$\begin{aligned}\Omega_{11i}^{(r)}(k) &= 2\bar{A}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{A}_i(k) \\ &\quad - P_i(k) + \bar{L}_i^T(k)\bar{L}_i(k) + \rho_i(k)\bar{E}(k) \\ \Omega_{21i}^{(r)}(k) &= \bar{B}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{A}_i(k) + \frac{1}{2}\tau_i(k)\tilde{C}_i(k) \\ \Omega_{22i}^{(r)}(k) &= 2\bar{B}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{B}_i(k) - \tau_i(k)I \\ \Omega_{31i}^{(r)}(k) &= \bar{D}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{A}_i(k) \\ \Omega_{32i}^{(r)}(k) &= \bar{D}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{B}_i(k) \\ \Omega_{33i}^{(r)}(k) &= 2\bar{D}_i^T(k)\bar{P}_i^{(r)}(k+1)\bar{D}_i(k) - \gamma^2 I \\ \bar{P}_i^{(r)}(k) &= \sum_{j=1}^s \lambda_{ij}^{(r)} P_j(k), \quad i, j = 1, \dots, s.\end{aligned}$$

Note that $P_i(k) = \text{diag}\{P_{1i}(k), P_{2i}(k)\}$ where $P_{1i}(k) \in \mathbb{R}^{n \times n}$ and $P_{2i}(k) \in \mathbb{R}^{n \times n}$. Noticing (29)–(30), by using Schur complement, S -procedure and some algebraic manipulations, we can obtain (26)–(27) and this completes the proof of the theorem. ■

Remark 2: Now, let us consider the uncertain discrete-time Markovian system (2) with known transition probability matrix $\hat{\Psi}$. In this case, the inequalities (26) and (27) reduce to

$$\Phi_i(k) = \begin{bmatrix} \Phi_{11i}(k) & * \\ \Phi_{21i}(k) & \Phi_{22i}(k) \end{bmatrix} < 0 \quad (31)$$

$$\Upsilon_i(k) = \begin{bmatrix} \Upsilon_{11i}(k) & * & * \\ \Upsilon_{21i}(k) & \Upsilon_{22i}(k) & * \\ \Upsilon_{31i}(k) & \Upsilon_{32i} & \Upsilon_{33i}(k) \end{bmatrix} \leq 0. \quad (32)$$

Theorem 2 provides a design scheme for a time-varying filter in the presence of unknown transition probability matrix in Case 1. Now we are going to consider the similar problem with unknown transition probability matrix in Case 2, and the following theorem is established along a similar line. Before stating the following theorem, let us denote

$$\begin{aligned}\hat{\Phi}_{21i}(k) &= \begin{bmatrix} \bar{f}\hat{P}_{1i}(k+1) & 0 \\ 0 & \sqrt{\beta(1-\beta)}\bar{N}_i(k) \\ \bar{\alpha}\hat{P}_{1i}(k+1) & 0 \\ 0 & \beta\bar{N}_i(k) \end{bmatrix} \\ \hat{\Phi}_{22i}(k) &= \text{diag} \left\{ -\hat{P}_{1i}(k+1), -\hat{P}_{2i}(k+1) \right. \\ &\quad \left. -\frac{1}{4}\hat{P}_{1i}(k+1), -\frac{1}{4}\hat{P}_{2i}(k+1) \right\} \\ \hat{\Upsilon}_{21i}(k) &= \begin{bmatrix} \hat{\Xi}_{4i}(k) & \hat{\Xi}_{5i}(k) \\ \hat{\Xi}_{6i}(k) & 0 \end{bmatrix} \\ \hat{\Upsilon}_{22i}(k) &= \text{diag} \left\{ -\hat{P}_{1i}(k+1), -\hat{P}_{2i}(k+1) \right. \\ &\quad \left. -\hat{P}_{1i}(k+1), -\hat{P}_{2i}(k+1) - \hat{P}_{1i}(k+1) \right\} \\ \hat{\Upsilon}_{31i}(k) &= \begin{bmatrix} 0 & \hat{\Xi}_{7i}(k) \\ \Xi_{8i} & 0 \end{bmatrix} \\ \bar{f} &= \sqrt{\bar{\alpha}(1-\bar{\alpha})} \\ \hat{\Upsilon}_{33i}(k) &= \text{diag} \left\{ -\hat{P}_{2i}(k+1), -\hat{P}_{1i}(k+1) \right. \\ &\quad \left. -\hat{P}_{2i}(k+1), -\xi_i I, -\xi_i I \right\}\end{aligned}$$

$$\begin{aligned}\hat{\Xi}_{4i}(k) &= \begin{bmatrix} \hat{P}_{1i}(k+1)A_i(k) & 0 \\ \bar{N}_i(k)K_1C_i(k) & \bar{M}_i(k) \end{bmatrix} \\ \hat{\Xi}_{5i}(k) &= \begin{bmatrix} 0 & \hat{P}_{1i}(k+1)D_{1i}(k) & 0 \\ \bar{N}_i(k) & \bar{N}_i(k)D_{2i}(k) & 0 \end{bmatrix} \\ \hat{\Xi}_{6i}(k) &= \begin{bmatrix} \hat{P}_{1i}(k+1)A_i(k) & 0 \\ \bar{N}_i(k)K_1C_i(k) & \bar{M}_i(k) \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Theorem 3: Consider system (2) with unknown transition probability matrix described in Case 2. Let $\gamma > 0$ be a given disturbance attenuation level. For given positive definite matrices $Q_i > 0$ ($i \in S$), assume that there exist families of positive definite matrices $\{P_{1i}(k)\}_{0 \leq k \leq N+1}$, $\{P_{2i}(k)\}_{0 \leq k \leq N+1}$ ($i = 1, 2, \dots, s$), families of positive scalars $\{\rho_i(k)\}_{0 \leq k \leq N}$, $\{\tau_i(k)\}_{0 \leq k \leq N}$, ($i = 1, 2, \dots, s$), $\{\xi_i\}_{i=1,2,\dots,s}$ and families of real-valued matrices $\{\bar{M}_i(k)\}_{0 \leq k \leq N}$, $\{\bar{N}_i(k)\}_{0 \leq k \leq N}$ and $\{L_{fi}(k)\}_{0 \leq k \leq N}$ satisfying the following RLMI:

$$\hat{\Phi}_i(k) = \begin{bmatrix} \Phi_{11i}(k) & * \\ \Phi_{21i}(k) & \Phi_{22i}(k) \end{bmatrix} < 0, \quad i = 1, 2, \dots, s, \quad r = 1, 2, \dots, \nu. \quad (33)$$

$$\hat{\Upsilon}_i(k) = \begin{bmatrix} \Upsilon_{11i}(k) & * & * \\ \Upsilon_{21i}(k) & \Upsilon_{22i}(k) & * \\ \Upsilon_{31i}(k) & \Upsilon_{32i} & \Upsilon_{33i}(k) \end{bmatrix} \leq 0, \quad i = 1, 2, \dots, s, \quad r = 1, 2, \dots, \nu \quad (34)$$

with the initial condition

$$\begin{bmatrix} P_{1i}(0) - \gamma^2 Q_i & \gamma^2 Q_i \\ \gamma^2 Q_i & P_{2i}(0) - \gamma^2 Q_i \end{bmatrix} \leq 0 \quad (35)$$

where $\hat{\Phi}_{21i}(k)$, $\hat{\Phi}_{22i}(k)$, $\hat{\Upsilon}_{21i}(k)$, $\hat{\Upsilon}_{22i}(k)$, $\hat{\Upsilon}_{31i}(k)$, $\hat{\Upsilon}_{33i}(k)$, $\hat{\Xi}_{4i}(k)$, $\hat{\Xi}_{5i}(k)$, and $\hat{\Xi}_{6i}(k)$ are defined previously, and $\Phi_{11i}(k)$, $\Upsilon_{11i}(k)$ and Υ_{32i} are the same as defined in Theorem 2, and if $\hat{\Psi}_k^i = \emptyset$, we take

$$\hat{P}_{1i}(k) = P_{1i}(k), \quad \hat{P}_{2i}(k) = P_{2i}(k), \quad i = 1, \dots, s,$$

otherwise

$$\begin{aligned}\hat{P}_{1i}(k) &= \left(\sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} \right)^{-1} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{1j}(k) \\ \hat{P}_{1i}(k) &= P_{1i}(k), \quad j \in \hat{\Psi}_{uk}^i \\ \hat{P}_{2i}(k) &= \left(\sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} \right)^{-1} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{2j}(k) \\ \hat{P}_{2i}(k) &= P_{2i}(k), \quad j \in \hat{\Psi}_{uk}^i.\end{aligned}$$

Then, there exists an n th order filter of the form (9) which ensures the \mathcal{H}_∞ performance constraint in (12), where $L_{fi}(k)$ is given as part of the RLMI solution and the other two filter parameters are given by

$$\begin{aligned}A_{fi}(k) &= \hat{P}_{2i}^{-1}(k+1)\bar{M}_i(k) \\ B_{fi}(k) &= \hat{P}_{2i}^{-1}(k+1)\bar{N}_i(k).\end{aligned}$$

Proof: Denote

$$\zeta_{ki} = \left(\sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} \right)^{-1} \quad (36)$$

It is clear that, for any $i \in S$, (31) and (32) can be rewritten as shown in the equation at the bottom of the page. Therefore, if

$$\begin{aligned} \hat{\Phi}_i(k) & \begin{cases} \hat{P}_{1i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{1j}(k), \\ \hat{P}_{2i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{2j}(k) \end{cases} < 0 \\ \hat{\Upsilon}_i(k) & \begin{cases} \hat{P}_{1i}(k) = P_{1i}(k), \\ \hat{P}_{2i}(k) = P_{2i}(k) \end{cases} < 0, j \in \hat{\Psi}_{uk}^i \\ \hat{\Upsilon}_i(k) & \begin{cases} \hat{P}_{1i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{1j}(k) \\ \hat{P}_{2i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{2j}(k) \end{cases} < 0 \\ \hat{\Upsilon}_i(k) & \begin{cases} \hat{P}_{1i}(k) = P_{1i}(k), \\ \hat{P}_{2i}(k) = P_{2i}(k) \end{cases} < 0, j \in \hat{\Psi}_{uk}^i \end{aligned}$$

we have $\Phi_i(k) < 0$ and $\Upsilon_i(k) < 0$ for any $i \in S$. This completes the proof. ■

Remark 3: Theorem 3 provides feasible solutions to the filter design problem for time-varying Markovian jump system (2) under partially unknown transition probabilities. Note that if $\hat{\Psi}_{uk}^i = \emptyset$ holds for any $i \in S$, i.e., all the transition probabilities are accessible, the corresponding results in (33)–(34) reduce to (31) and (32). Similarly, when $\hat{\Psi}_k^i = \emptyset$ holds for any $i \in S$, i.e., all the transition probabilities are inaccessible, Theorem 3 is still valid at the cost of the incremental conservatism. More specifically, the more known entries in the transition probability matrix, the less conservatism of the results we would have.

Based on the Theorem 2 and Theorem 3, we suggest the following Robust H_∞ Filter Design (RHFD) algorithm involving recursive LMIs conditions.

Algorithm RHFD:

- Step 1. Given the \mathcal{H}_∞ performance index γ , positive definite matrices Q_i ($i = 1, 2, \dots, s$) and the state initial conditions $x(0)$ and $\hat{x}(0)$. Select the initial values for matrices $\{P_{1i}(0)$ and $P_{2i}(0)\}$ which satisfy the condition (28) and set $k = 0$.
- Step 2. Obtain the positive matrices $P_{1i}(k+1)$ and $P_{2i}(k+1)$, and matrices $\bar{M}_i(k)$, $\bar{N}_i(k)$, $L_{fi}(k)$ for the sampling instant k by solving the RLMI (26)–(27) or (33)–(34), respectively, with known parameters $P_{1i}(k)$ and $P_{2i}(k)$.

- Step 3. Derive the other two filter parameter matrices $A_{fi}(k)$ and $B_{fi}(k)$ by solving (29), and set $k = k+1$.
- Step 4. If $k < N$, then go to Step 2, otherwise exit.

V. ILLUSTRATIVE EXAMPLES

In this section, we present two simulation examples to illustrate the usefulness and flexibility of the time-varying filter design method developed in this paper. Consider a class of uncertain discrete stochastic nonlinear time-varying Markovian jump systems with sensor saturation in the form (2).

Example 1: Consider Case 1 where the transition probability matrix of the Markov process is unknown but it resides in a polytope with the following two vertices:

$$\hat{\Psi}^{(1)} = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}, \quad \hat{\Psi}^{(2)} = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}.$$

Suppose that the system involves two modes, and the system data are given as follows:

Mode 1:

$$\begin{aligned} A_1(k) &= \begin{bmatrix} 0.2 & 0.2 \sin(k) \\ 1.1 \sin(5k) & 0.5 \end{bmatrix} \\ D_{11}(k) &= \begin{bmatrix} 0.1 \sin(3k) \\ -0.3 \end{bmatrix} \\ H_{11} &= \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} \\ D_{21}(k) &= -0.3 \sin(3k) \\ H_{21} &= 0.2, \\ L_1(k) &= [0.3 \sin(2k) \quad 0.7] \\ N_1 &= [0 \quad 0.5] \\ C_1(k) &= [0.9 \quad 0.5 \sin(5k)], \quad |F_1(k)| \leq 1 \end{aligned}$$

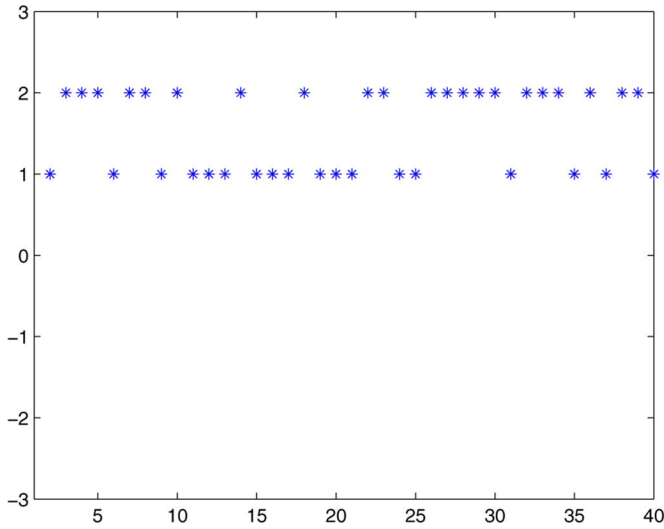
$\sigma(y_{s1}(k))$ is a saturation function described as follows:

$$\sigma(y_{s1}(k)) = \begin{cases} \sigma(y_{s1}(k)) = y_{s1}(k), & \text{if } -V_{ys1j,\max} \leq y_{s1}(k) \leq V_{ys1j,\max}; \\ \sigma(y_{s1}(k)) = V_{ys1j,\max}, & \text{if } y_{s1}(k) > V_{ys1j,\max}; \\ \sigma(y_{s1}(k)) = -V_{ys1j,\max}, & \text{if } y_{s1}(k) < -V_{ys1j,\max}; \end{cases}$$

Mode 2:

$$\begin{aligned} A_2(k) &= \begin{bmatrix} 0.3 \sin(k) & 0.1 \\ 1.3 & 0.5 \sin(5k) \end{bmatrix}, \\ D_{12}(k) &= \begin{bmatrix} 0.1 \\ 0.4 \sin(3k) \end{bmatrix}, H_{12} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \Phi_i(k) &= \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} \hat{\Phi}_i(k) \begin{cases} \hat{P}_{1i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{1j}(k) \\ \hat{P}_{2i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{2j}(k) \end{cases} + \sum_{j \in \hat{\Psi}_{uk}^i} \lambda_{ij} \hat{\Phi}_i(k) \begin{cases} \hat{P}_{1i}(k) = P_{1i}(k) \\ \hat{P}_{2i}(k) = P_{2i}(k) \end{cases} \\ \Upsilon_i(k) &= \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} \hat{\Upsilon}_i(k) \begin{cases} \hat{P}_{1i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{1j}(k) \\ \hat{P}_{2i}(k) = \zeta_{ki} \sum_{j \in \hat{\Psi}_k^i} \lambda_{ij} P_{2j}(k) \end{cases} + \sum_{j \in \hat{\Psi}_{uk}^i} \lambda_{ij} \hat{\Upsilon}_i(k) \begin{cases} \hat{P}_{1i}(k) = P_{1i}(k) \\ \hat{P}_{2i}(k) = P_{2i}(k) \end{cases} \end{aligned}$$

Fig. 1. Random mode $r(k)$.

$$\begin{aligned}
 D_{22}(k) &= -0.2 \sin(4k), \quad H_{22} = 0.1, \\
 L_2(k) &= [0.4 \sin(2k) \quad 0.2], \quad N_2 = [0 \quad 0.5], \\
 C_2(k) &= [1.3 \quad 0.2 \sin(k)], \quad |F_2(k)| \leq 1 \\
 \sigma(y_{s2}(k)) &= \begin{cases} \sigma(y_{s2}(k)) = y_{s2}(k), & \text{if } -V_{ys2j,\max} \leq y_{s2}(k) \leq V_{ys2j,\max}; \\ \sigma(y_{s2}(k)) = V_{ys2j,\max}, & \text{if } y_{s2}(k) > V_{ys2j,\max}; \\ \sigma(y_{s2}(k)) = -V_{ys2j,\max}, & \text{if } y_{s2}(k) < -V_{ys2j,\max}; \end{cases}
 \end{aligned}$$

and the nonlinear functions $f(k, x(k))$ and $g(k, x(k))$ are selected as

$$\begin{aligned}
 f(k, x(k)) &= \begin{bmatrix} \frac{0.2x_1(k)}{2x_2^2(k)+1} & 0.1x_1(k) \sin(x_2(k)) \end{bmatrix}^T \\
 g(k, x(k)) &= 0.2x_1(k) \sin(x_2(k)).
 \end{aligned}$$

It is easy to see that the constraint (4) can be met with $\varepsilon_1(k) = \varepsilon_2(k) = 1$ and $E_1(k) = E_2(k) = \text{diag}\{0.2, 0.15\}$. In this example, the saturation values are taken as $V_{ys11} = V_{ys21} = 0.06$ and $K = 0.2$, $K_1 = 0.8$. The state initial value $x(0) = [0.2 \quad -0.5]^T$, $\hat{x}(0) = [-0.2 \quad -0.16]^T$. The exogenous disturbance input $w(k)$ is supposed to be a random noise uniformly distributed over $[-0.5 \quad 0.5]$ and the probabilities are assumed to be $\bar{\alpha} = \bar{\beta} = 0.9$. Set $\gamma = 0.5$ and let $Q_1 = Q_2 = \text{diag}\{1, 1\}$. Choose the parameters' initial values to satisfy (28).

Consider the real transition probability matrix as

$$\hat{\Psi} = \begin{bmatrix} 0.56 & 0.44 \\ 0.42 & 0.58 \end{bmatrix}$$

which means that $\psi_1 = 0.4$ and $\psi_2 = 0.6$ in (1). According to the robust H_∞ filter design algorithm (RHFD), the RLMI in Theorem 2 can be solved recursively subject to given initial conditions and prespecified performance indices.

The simulation results are shown in Figs. 1–3, where Fig. 1 plots one of the possible realizations of the Markovian jumping mode $r(k)$. Under this mode sequence, the estimation error $\bar{z}(k)$ is depicted in Fig. 2. Fig. 3 shows the sensor output. Note that

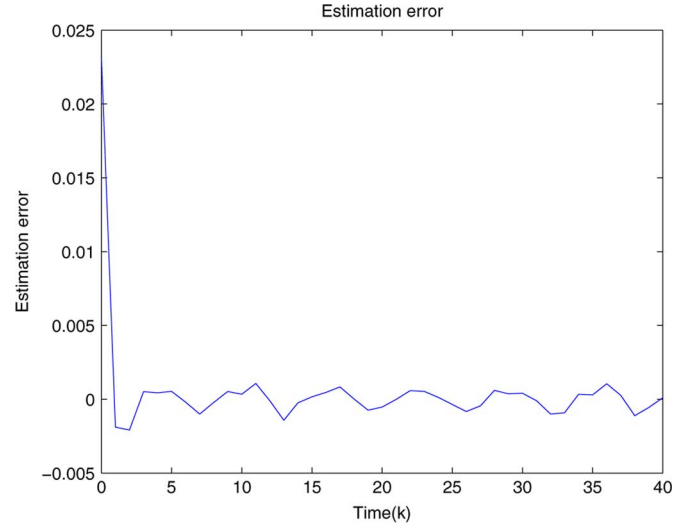


Fig. 2. Estimation error.

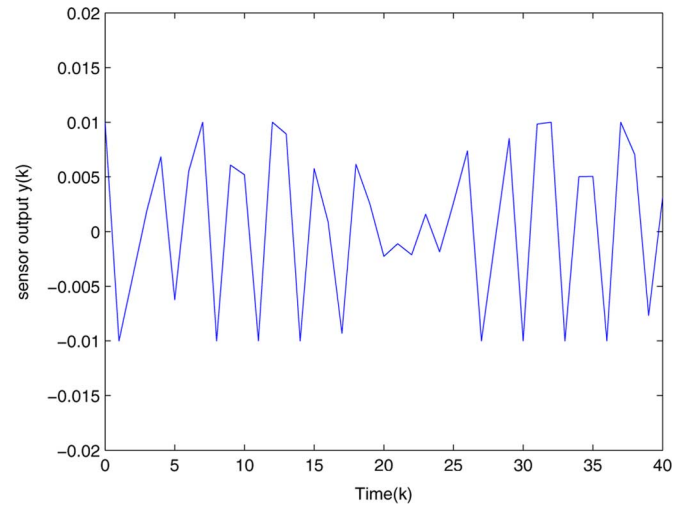


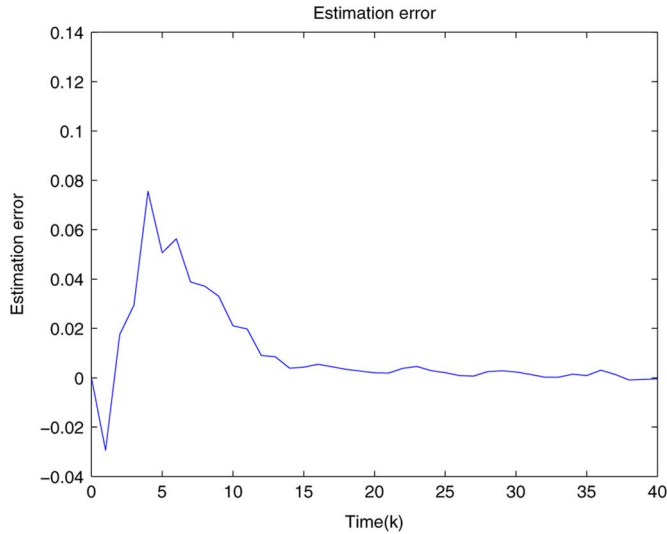
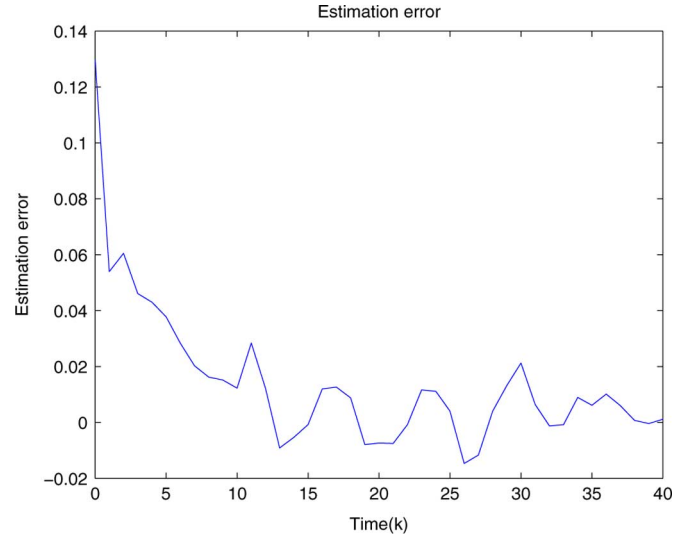
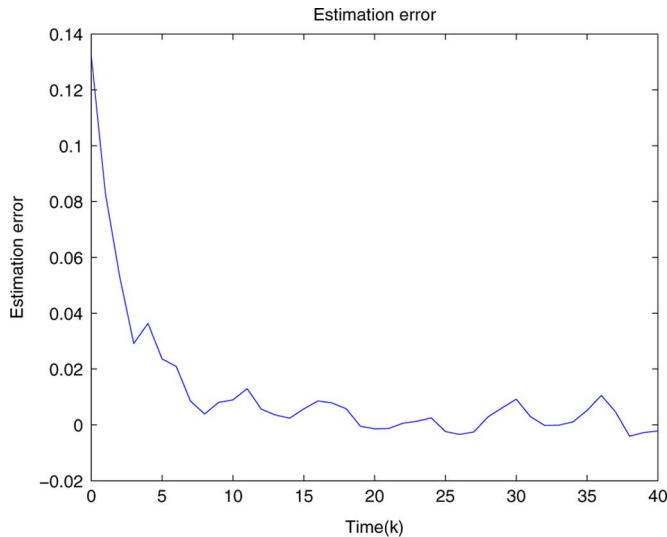
Fig. 3. Sensor output.

the sensor outputs is saturated. The simulation has confirmed that the designed filter performs very well.

Example 2: Consider *Case 2* where some elements in the transition probability matrix of the Markov process are unknown and the possible three cases for $\hat{\Psi}$ are given as follows:

$$\begin{aligned}
 \hat{\Psi}_1 &= \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}, \\
 \hat{\Psi}_2 &= \begin{bmatrix} 0.6 & 0.4 \\ ? & ? \end{bmatrix}, \quad \hat{\Psi}_3 = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}
 \end{aligned}$$

where $\hat{\Psi}_1$ (respectively, $\hat{\Psi}_2$, $\hat{\Psi}_3$) shows that the elements in transition probabilities matrix are completely known (respectively, partially known or completely unknown), and the other parameters of the discrete stochastic nonlinear time-varying system (2) are the same as in *Example 1*. Similarly, according to the robust H_∞ filter design algorithm (RHFD), the RLMI in Theorem 3 can be solved recursively subject to given initial conditions and prespecified performance indices. The corresponding simulation results for the estimation error in these three cases

Fig. 4. Estimation error of $\hat{\Psi}_1$ case.Fig. 6. Estimation error of $\hat{\Psi}_3$ case.Fig. 5. Estimation error of $\hat{\Psi}_2$ case.

are given in Figs. 4–6, respectively. Again, it can be seen that the more known entries in the transition probability matrix we have, the less conservatism of the condition there would be.

VI. CONCLUSION

In this paper, the robust \mathcal{H}_∞ filtering problem has been investigated for time-varying Markovian jump systems with randomly occurring nonlinearities and sensor saturation. The considered transition probability matrix includes the case with polytopic uncertainties and the case with partially unknown transition probabilities, respectively. Also, the case with completely known or completely unknown transition probabilities have been studied as two special cases. The randomly occurring nonlinearities have been modeled by the Bernoulli distributed white sequences with known conditional probabilities. Sufficient conditions have been derived for the filtering augmented system under consideration to satisfy the \mathcal{H}_∞ performance constraint. The corresponding robust \mathcal{H}_∞ filters have been designed by solving sets of RLMI. Two numerical simulation

examples have been used to demonstrate the effectiveness of the filtering technology presented in this paper.

REFERENCES

- [1] C. E. De Souza, A. Trofino, and K. A. Barbosa, "Mode-independent \mathcal{H}_∞ filters for Markovian jump linear systems," *IEEE Trans. Autom. Control*, vol. 51, no. 11, pp. 1837–1841, 2006.
- [2] C. E. De Souza, D. F. Coutinho, and K. A. Barbosa, " \mathcal{H}_∞ filtering for a class of uncertain Markov jump nonlinear systems," in *Proc. 45th IEEE Conf. Decision Control*, 2006, pp. 5531–5536.
- [3] H. Dong, Z. Wang, D. W. C. Ho, and H. Gao, "Variance-constrained H_∞ filtering for nonlinear time-varying stochastic systems with multiple missing measurements: The finite-horizon case," *IEEE Trans. Signal Process.*, vol. 58, no. 5, pp. 2534–2543, 2010.
- [4] H. Dong, Z. Wang, and H. Gao, "Robust H_∞ filtering for a class of nonlinear networked systems with multiple stochastic communication delays and packet dropouts," *IEEE Trans. Signal Process.*, vol. 58, no. 4, pp. 1957–1966, 2010.
- [5] K. Kameyama and A. Ohsum, "Subspace-based prediction of linear time-varying stochastic systems," *Automatica*, vol. 43, no. 12, pp. 2009–2021, 2007.
- [6] H. Liu, F. Sun, K. He, and Z. Sun, "Design of reduced-order \mathcal{H}_∞ filter for Markovian jumping systems with time delay," *IEEE Trans. Circuits Syst. II*, vol. 51, no. 11, pp. 1837–1841, 2006.
- [7] Z. Fei, H. Gao, and P. Shi, "New results on stabilization of Markovian jump systems with time delay," *Automatica*, vol. 45, no. 10, pp. 2300–2306, 2009.
- [8] H. Gao, Y. Zhao, J. Lam, and K. Chen, " H_∞ fuzzy filtering of nonlinear systems with intermittent measurements," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 2, pp. 291–300, 2009.
- [9] H. Gao, T. Chen, and J. Lam, "A new delay system approach to network based control," *Automatica*, vol. 44, no. 1, pp. 39–52, 2008.
- [10] G. Garcia, S. Tarbouriech, and J. M. Gomes da Silva, Jr., "Dynamic output controller design for linear systems with actuator and sensor saturation," in *Proc. 2007 Amer. Control Conf.*, 2007, pp. 5834–5839.
- [11] Y. Niu, T. Jia, X. Wang, and F. Yang, "Output-feedback control design for NCSs subject to quantization and dropout," *Inf. Sci.*, vol. 179, no. 21, pp. 3804–3813, 2009.
- [12] P. Shi, M. Mahmoud, S. K. Nguang, and A. Ismail, "Robust filtering for jumping systems with mode-dependent delays," *Signal Process.*, vol. 86, no. 1, pp. 140–152, 2006.
- [13] P. Shi, E. K. Boukas, and R. K. Agarwal, "Kalman filtering for continuous-time uncertain systems with Markovian jumping parameters," *IEEE Trans. Autom. Control*, vol. 44, no. 8, pp. 1592–1597, 1999.
- [14] M. H. Terra, J. Y. Ishihara, and G. Jesus, "Information filtering and array algorithms for discrete-time Markovian jump linear systems," *IEEE Trans. Autom. Control*, vol. 54, no. 1, pp. 158–162, 2009.
- [15] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust H_∞ filtering for stochastic time-delay systems with missing measurements," *IEEE Trans. Signal Process.*, vol. 54, no. 7, pp. 2579–2587, 2006.

- [16] Z. Wang, D. W. C. Ho, H. Dong, and H. Gao, "Robust \mathcal{H}_∞ finite-horizon control for a class of stochastic nonlinear time-varying systems subject to sensor and actuator saturations," *IEEE Trans. Autom. Control*, vol. 55, no. 7, pp. 1716–1722, 2010.
- [17] Z. Wang, Y. Wang, and Y. Liu, "Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time-delays," *IEEE Trans. Neural Netw.*, vol. 21, no. 1, pp. 11–25, 2010.
- [18] G. Wei, Z. Wang, H. Shu, and J. Fang, "A delay-dependent approach to \mathcal{H}_∞ filtering for stochastic delayed jumping systems with sensor nonlinearities," *Int. J. Control*, vol. 80, no. 6, pp. 885–897, 2007.
- [19] Y. Xiao, Y. Cao, and Z. Lin, "Robust filtering for discrete-time systems with saturation and its application to transmultiplexers," *IEEE Trans. Signal Process.*, vol. 52, no. 5, pp. 1266–1277, 2004.
- [20] N. Xiao, L. Xie, and M. Fu, "Kalman filtering over unreliable communication networks with bounded Markovian packet dropouts," *Int. J. Robust Nonlinear Control*, vol. 19, no. 16, pp. 1770–1786, 2009.
- [21] L. Wu, P. Shi, H. Gao, and C. Wang, " \mathcal{H}_∞ filtering for 2-D Markovian jump systems," *Automatica*, vol. 44, no. 7, pp. 1849–1858, 2008.
- [22] J. Xiong, J. Lam, H. Gao, and D. W. C. Ho, "On robust stabilization of Markovian jump systems with uncertain switching probabilities," *Automatica*, vol. 41, no. 5, pp. 897–903, 2005.
- [23] J. Xiong and J. Lam, "Fixed-order robust \mathcal{H}_∞ filter design for Markovian jump systems with uncertain switching probabilities," *IEEE Trans. Signal Process.*, vol. 54, no. 4, pp. 1421–1430, 2006.
- [24] F. Yang and Y. Li, "Set-membership filtering for systems with sensor saturation," *Automatica*, vol. 45, no. 8, pp. 1896–1902, 2009.
- [25] F. Yang and Y. Li, "Set-membership filtering for discrete-time systems with nonlinear equality constraints," *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2480–2486, 2009.
- [26] W. Zhang, Y. Huang, and H. Zhang, "Stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control for discrete-time systems with state and disturbance dependent noise," *Automatica*, vol. 43, no. 3, pp. 513–521, 2007.
- [27] H. Zhang, G. Feng, G. Duan, and X. Lu, " \mathcal{H}_∞ filtering for multiple-time-delay measurements," *IEEE Trans. Signal Process.*, vol. 54, no. 5, pp. 1681–1688, 2006.
- [28] H. Zhang, G. Feng, and X. Lu, "White noise \mathcal{H}_∞ fixed-lag smoothing for continuous time systems," *Signal Process.*, vol. 87, no. 3, pp. 432–440, 2007.
- [29] L. Zhang, E. Boukas, and J. Lam, "Analysis and synthesis of Markov jump linear systems with time-varying delays and partially known transition probabilities," *IEEE Trans. Autom. Control*, vol. 53, no. 10, pp. 2458–2464, 2008.
- [30] L. Zhang and E. Boukas, "Mode-dependent \mathcal{H}_∞ filtering for discrete-time Markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 6, pp. 1462–1467, 2009.
- [31] X. Zhang and Y. Zheng, "Nonlinear \mathcal{H}_∞ filtering for interconnected Markovian jump systems," *J. Syst. Eng. Electron.*, vol. 17, no. 1, pp. 138–146, 2006.
- [32] D. H. Zhou and P. M. Frank, "Strong tracking filtering of nonlinear time-varying stochastic systems with coloured noise: Application to parameter estimation and empirical robustness analysis," *Int. J. Control*, vol. 65, no. 2, pp. 295–307, 1996.



Hongli Dong (M'11) received the B.E. degree in computer science and technology from Heilongjiang Institute of Science and Technology, Harbin, China, in 2000 and the M.E. degree in control theory and engineering from Northeast Petroleum University, Daqing, China, in 2003. She is currently working toward the Ph.D. degree in control science and engineering from the Harbin Institute of Technology, Harbin.

From July 2009 to January 2010, she was a Research Assistant in the Department of Applied Mathematics, the City University of Hong Kong. From October 2010 to January 2011,

she was a Research Assistant in the Department of Mechanical Engineering, the University of Hong Kong. She is now a Visiting Scholar in the Department of Information Systems and Computing, Brunel University, U.K. She is currently a Lecturer at Northeast Petroleum University. Her current research interests include robust control and networked control systems.

Dr. Dong is an active reviewer for many international journals.



Zidong Wang (SM'03) was born in Jiangsu, China, in 1966. He received the B.Sc. degree in mathematics from Suzhou University, Suzhou, China, in 1986 and the M.Sc. degree in applied mathematics and the Ph.D. degree in electrical and computer engineering both from Nanjing University of Science and Technology, Nanjing, China, in 1990 and 1994, respectively.

He is currently a Professor of Dynamical Systems and Computing at Brunel University, U.K. His research interests include dynamical systems,

signal processing, bioinformatics, and control theory and applications. He has published more than 120 papers in refereed international journals.

Dr. Wang is currently serving as an Associate Editor for 12 international journals, including the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, the IEEE TRANSACTIONS ON NEURAL NETWORKS, the IEEE TRANSACTIONS ON SIGNAL PROCESSING, the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART C, and the IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY.



Daniel W. C. Ho (SM'06) received the B.Sc., M.Sc., and Ph.D. degrees (First Class) in mathematics from the University of Salford, U.K., in 1980, 1982, and 1986, respectively.

From 1985 to 1988, he was a Research Fellow in the Industrial Control Unit, University of Strathclyde, Glasgow, Scotland. In 1989, he joined the Department of Mathematics, City University of Hong Kong, where he is currently a Professor. His research interests include H-infinity control theory, adaptive neural wavelet identification, nonlinear control theory, complex network, networked control system, and quantized control.

Dr. Ho is currently serving as an Associate Editor for the *Asian Journal of Control*.



Huijun Gao (SM'10) was born in Heilongjiang Province, China, in 1976. He received the M.S. degree in electrical engineering from Shenyang University of Technology, Shenyang, China, in 2001 and the Ph.D. degree in control science and engineering from Harbin Institute of Technology, Harbin, China, in 2005.

He is currently a Professor at Harbin Institute of Technology, China. His research interests include network-based control, robust control/filter theory, model reduction, time-delay systems, and multidimensional systems, and their applications.

Dr. Gao is an Associate Editor or member of editorial board for several journals, such as *Automatica*, the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART B, the IEEE TRANSACTIONS ON INDUSTRIAL ELECTRONICS, the IEEE TRANSACTIONS ON FUZZY SYSTEMS, and the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS.